Clothoid Based Spline-RRT with Bézier Approximation

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Outline of the Presentation

1. Problem formulation
2. Overview of the relevant research
3. Methodology
4. Results and discussions
Problem Formulation

- **Planar Path**
  A piecewise curve as a superset of $n$ segments

  $$\mathcal{P}_s = \bigcup_{i=1}^{n} \mathcal{P}_i \quad \text{and} \quad \mathcal{P}_i := \{ \mathbf{p}_i \in \mathbb{R}^2 | \mathbf{p}_i = \mathbf{p}_{i,p}(t) \ \forall t \in I_i = [t_{i,0}, t_{i,1}] \subseteq \mathbb{R} \}$$

- **Path Smoothing**
  - Parametric continuity
  - Geometric Continuity
Problem Formulation

- Curvature continuity

![Graph showing curvature continuity](image)
Problem Formulation

- **Clothoid**

  A curve whose curvature changes linearly with its curve length (Euler Spiral)

  **Advantage**: Shortest path satisfying Maximum Principle (optimal control theory)

  \[ F(s) = \begin{pmatrix} x_0 + \int_0^s \cos \left( \theta_0 + \kappa_0 u + \frac{\sigma u^2}{2} \right) \, du \\ y_0 + \int_0^s \sin \left( \theta_0 + \kappa_0 u + \frac{\sigma u^2}{2} \right) \, du \end{pmatrix} \]

  \[ \kappa(s) = \kappa_0 + \sigma s, \quad \kappa_0, \sigma \in \mathbb{R} \]

  **Disadvantage**: No closed form due to Fresnel integrals

  \[ C(t) = \int_0^t \cos \left( \frac{\pi u^2}{2} \right) \, du \quad \text{and} \quad S(t) = \int_0^t \sin \left( \frac{\pi u^2}{2} \right) \, du \]
Relevant Research

- **Online Approximation**
  - RBC Algorithm (Montés, N., 2008)
  - Circular Interpolation (Brezak, M., 2014)

- **Offline Approximation**

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<th>Method</th>
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<td>1</td>
<td>Continuous function approximation (Wang, Lazhu Z., 2001)</td>
<td>Degree can be 26th order</td>
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<td>2</td>
<td>$C^2$ Hermite interpolation via s-power series (Sánchez-Reyes, 2003)</td>
<td>Complicated coefficients calculation</td>
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<td>3</td>
<td>$G^3$ Bézier approximation with numerical search (Cross, 2012; L Lu., 2013)</td>
<td>Numerical search procedure is expensive; not robust</td>
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<tr>
<td>4</td>
<td>$G^{2+}$ deterministic approximation (Cross, 2015)</td>
<td>Not accurate due to linear approximation</td>
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Note: 1, 2, 3, 4 can only deal with **unit-lenth** clothoids
Methodology

Elementary Clothoid $F_\varepsilon(s)$

Basic Clothoid $F_L(s)$

General Clothoid $F(s)$

Lookup Table
Methodology

(a) Bézier curves \( \mathbf{B}_{\varepsilon,i} (i = 1, 2, \ldots, 7) \) as primitives
(b) The Bézier approximation to the basic clothoid

\[
\mathbf{B}_\varepsilon (t) = \sum_{i=0}^{5} \binom{5}{i} (1 - t)^{5-i} t^i \mathbf{V}_i, \quad t \in [0, 1].
\]

\[
\kappa_b'(s)|_{s=0} = \kappa_c'(s)|_{s=0} = \kappa_c'(s)|_{s=u}
\]

**G^3 Continuity Constraints**
Coordinate (Position) Error

- Continuous: Pythagorean error in the Cornu plane (1,2)
  \[ \epsilon_p = \| B(s) - F_L(s) \| \]

- Discrete: Euclidean distance (3)
  \[ \epsilon_c = \sqrt{(x_e - x_a)^2 + (y_e - y_a)^2} \]

Orientation and Curvature Errors

- **Orientation Error**
  \[ \epsilon_\theta = \| \theta_b(s) - \theta_c(s) \| \]

- **Curvature Error (4, 5, 6, 7)**
  \[ \epsilon_\kappa = \max_{s \in [0,l]} \epsilon_\kappa(s) = \max_{s \in [0,l]} \frac{|\kappa_b(s) - \kappa_c(s)|}{\max \{ |\kappa_c(s)|, 1 \}} \]

Circular Interpolation (Brezak, M., 2014)

\[ \sigma = 1.0, \Delta s = 1.0, \kappa_0 = 0.0, \theta_0 = 0.0, (x_0, y_0) = (0.0, 0.0) \]
RBC Algorithm (Montés, N., 2008)

9-th order RBC

\[ P(\gamma) = \frac{\sum_{k=0}^{N} w_k \cdot c_k \cdot B_k}{\sum_{k=0}^{N} w_k \cdot B_k} \]
Accuracy

\( \sigma = 1.0, s = 6.0, \kappa_0 = 0.0, \theta_0 = 0.0, (x_0, y_0) = (0.0, 0.0) \)

Uniform: N=30
Nonuniform: N=19
Efficiency

- Retrieve 100 Points

Mean execution time ($\mu s$):

- RBC Algorithm: 1477.5
- Circular Interpolation: 199.5
- Proposed Approach: 305.0
Scalability

Position error with respect to clothoid scaling

\[ \epsilon_p(s) = \sqrt{\frac{\sigma}{\sigma_\mathcal{C}}} \epsilon_{p,\mathcal{C}}(s_\mathcal{C}). \]
Differential System

- Extension of Dubins car (8)

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\kappa}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta \\
\sin \theta \\
\kappa \\
0
\end{pmatrix} v + \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} \sigma,
\]

\[
\kappa_{\text{max}} = \frac{\tan \phi}{L_w},
\]

\[
\sigma_{\text{max}} = \frac{\dot{\phi}}{L_w \cos^2 \phi},
\]

Feasible Path: G² curvature continuous with curvature derivative (sharpness) bounded

Local Path

- Case 1: A pair of clothoids, a circular arc and line segments

\[ h = y_c \left( \frac{\kappa_{\text{max}}}{\sigma_{\text{max}}} \right) + \frac{1}{\kappa_{\text{max}}} \cos \left( \frac{\kappa_{\text{max}}^2}{2\sigma_{\text{max}}} \right). \]

\[ d = x_c \left( \frac{\kappa_{\text{max}}}{\sigma_{\text{max}}} \right) + h \tan \left( \frac{|\delta|}{2} \right) - \frac{1}{\kappa_{\text{max}}} \sin \left( \frac{\kappa_{\text{max}}^2}{2\sigma_{\text{max}}} \right). \]
Local Path

- Case 1

\[
\begin{pmatrix}
  x_{c1}(s) \\
  y_{c1}(s)
\end{pmatrix}
= R(\delta_0)
\begin{pmatrix}
  x_c(s) \\
  \text{sgn}(\delta)y_c(s)
\end{pmatrix}
- d\hat{e}_1 + W_2,
\]

\[
\begin{pmatrix}
  x_{c2}(s) \\
  y_{c2}(s)
\end{pmatrix}
= R(\delta_0 + \delta)
\begin{pmatrix}
  -x_c(s) \\
  \text{sgn}(\delta)y_c(s)
\end{pmatrix}
+ dR(\delta)\hat{e}_1 + W_2.
\]

\[
\begin{pmatrix}
  x_O \\
  y_O
\end{pmatrix}
= R(\delta_0)
\begin{pmatrix}
  d - h\tan\left(\frac{|\delta|}{2}\right) \\
  \text{sgn}(\delta)h
\end{pmatrix}
- d\hat{e}_1 + W_2.
\]

\[
\begin{cases}
  \left[\delta_0 + \frac{1}{2}(\delta_{\text{min}} - \pi), \delta_0 + |\delta| - \frac{1}{2}(\delta_{\text{min}} + \pi) \right], & \text{if } \delta > \delta_{\text{min}} \\
  \left[\delta_0 - |\delta| + \frac{1}{2}(\delta_{\text{min}} + \pi), \delta_0 - \frac{1}{2}(\delta_{\text{min}} - \pi) \right], & \text{if } \delta < -\delta_{\text{min}}
\end{cases}
\]
Local Path

- Case 2: Degenerated case with small deflection

\[ \delta = x_c \left( \sqrt{\frac{|\delta|}{\sigma_{\text{max}}}} \right) + y_c \left( \sqrt{\frac{|\delta|}{\sigma_{\text{max}}}} \right) \tan \left( \frac{|\delta|}{2} \right). \]
Comparison

- Bezier based planner (9)

Max solve time: 200000
Max curvature: 10
Diagonal length of environment: 15.8053
Max distance each step: 0.4
Robot dimensions: (0.127, 0.0635)
Length of robot: 0.00635
Start state 0: (-1.15, -1.1)
Start state 1: (-1.14984, -0.9)
goal state: (1.16, 1.1)
RRT: Created 1876 states
Solution found in 1.256377 seconds
Geometric path with 22 states

[9] 2014. Optimal path planning based on spline-RRT star for fixed-wing UAVs operating in three-dimensional environments
Comparison

- Proposed method

Max solve time: 200000
Max curvature: 10
Diagonal length of environment: 15.8053
Max distance each step: 0.4
Robot dimensions: (0.127, 0.0635)
Length of robot: 0.00635
Start state 0: (-1.15, -1.1)
Start state 1: (-1.14984, -0.9)
Goal state: (1.16, 1.1)
RRT: Created 522 states
Solution found in 0.511055 seconds
Geometric path with 20 states
Thank You!